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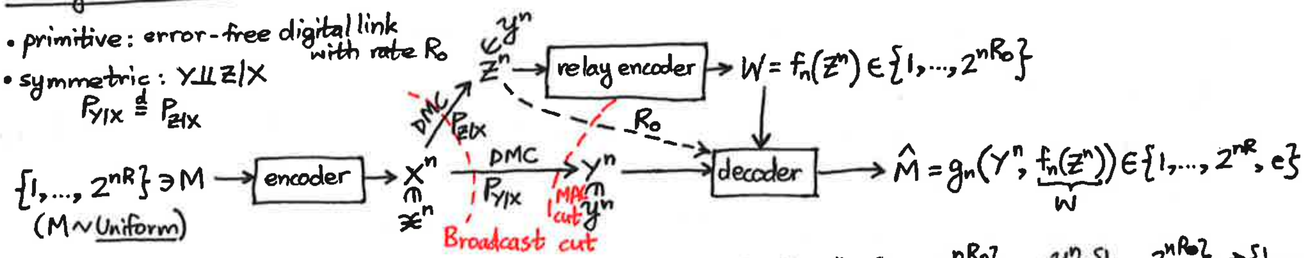
INFORMATION INEQUALITY:

① Relay Channel:

- primitive: error-free digital link with rate R_0
- symmetric: $Y \perp\!\!\!\perp Z \mid X$, $P_{Y|X} \overset{d}{=} P_{Z|X}$



- $\underline{(2^{nR}, n)\text{-code}}$: $\Big($message set $\{1,\dots,2^{nR}\}$, $\underset{\uparrow \text{encoder}}{x^n} : \{1,\dots,2^{nR}\} \to \mathcal{X}^n$, $\underset{\uparrow \text{relay encoder}}{f_n} : \mathcal{Y}^n \to \{1,\dots,2^{nR_0}\}$, $\underset{\uparrow \text{decoder}}{g_n} : \mathcal{Y}^n \times \{1,\dots,2^{nR_0}\} \to \{1,\dots,2^{nR}, e\}\Big)$
- Probability of error: $P_e^{(n)} = \mathbb{P}(\hat{M} \neq M)$ ← average
- Rate R is achievable if $\exists$ sequence of $(2^{nR}, n)$-codes with $P_e^{(n)} \to 0$ as $n \to \infty$.
- Capacity: $C(R_0) \triangleq \sup\{R \geq 0 : R \text{ achievable}\}$.

② Cutset Bound: (Cover-ElGamal '79)

choice of codes determine P_{X^n} from P_M via $x^n(\cdot)$

For a primitive relay channel, if rate R is achievable, then $\exists P_X$ such that:

↳ construct from P_{X_i}'s by averaging

$$R \leq I(X;Y,Z), \quad \text{[Broadcast bound]}$$
$$R \leq I(X;Y) + R_0. \quad \text{[MAC bound]}$$

Remarks:

Tightness

1. $C(0) = \max_{P_X} I(X;Y)$ and $C(\infty) = \max_{P_X} I(X;Y,Z)$. [Cover: Find min R_0 s.t. $C(R_0) = \max_{P_X} I(X;Y,Z)$.]

   ↑ MAC equality ↑ Broadcast equality (true for $R_0 \geq \log(|\mathcal{Y}|)$)

2. Reverse "Physically" Degraded Primitive Channel: (not symmetric)

   $P_{Y,Z|X} = P_{Y|X} P_{Z|Y}$ $[Z \perp\!\!\!\perp X \mid Y]$ $X \to Y \to Z$

   $I(X;Y,Z) = H(X) - H(X|Y,Z) = H(X) - H(X|Y) = I(X;Y)$

   $\Rightarrow R \leq I(X;Y)$ [Broadcast bound]

   So, $R = \max_{P_X} I(X;Y)$ is the relay channel capacity [for achievability, don't use relay].

First non-tight example

3. Zhang '88: Primitive "Stochastically" Degraded Channel (not symmetric)

   Assume $Y \perp\!\!\!\perp Z \mid X$, $P_{Y|X} = P_{Z|X} \circ Q_{Y|Z}$ for some $Q_{Y|Z}$. Then:

   $$R_0 > \max_{P_X : I(X;Y) = C_{XY}} I(X;Z) - C_{XY} \Rightarrow C(R_0) < C_{XY} + R_0, \text{ where } C_{XY} = \max_{P_X} I(X;Y).$$

   Note: Known that: $R_0 \leq \max_{P_X : I(X;Y) = C_{XY}} I(X;Z) - C_{XY} \Rightarrow C(R_0) = C_{XY} + R_0$. [Intuition: We can send $C_{XY} + R_0$ rate to Z, which will send the R_0 part to decoder.]

   Note: $\max_{P_X : I(X;Y) = C_{XY}} I(X;Z) - C_{XY} < R_0 < \max_{P_X : I(X;Y) = C_{XY}} I(X;Y,Z) - C_{XY} \Rightarrow$ Cutset Bound is $R \leq C_{XY} + R_0$ BUT no equality due to Zhang's result.

Proof: $nR = H(M) = I(M;\hat{M}) + H(M|\hat{M}) \leq I(M;\hat{M}) + n\mathcal{J}_n$, $\mathcal{J}_n \to 0$ as $n \to \infty$

[Fano: $M \to X^n \to (Y^n, Z^n) \to \hat{M}$, $H(M|\hat{M}) \leq H(P_e^{(n)}) + P_e^{(n)} nR = n \underbrace{\left(\frac{H(P_e^{(n)})}{n} + P_e^{(n)} R\right)}_{\to 0 \text{ as } n \to \infty}$]

$\overset{\text{DPI}}{\Rightarrow} nR \leq I(X^n; Y^n, W) + n\mathcal{J}_n$

Broadcast bound:

$nR \leq I(X^n; Y^n, Z^n) + n\mathcal{J}_n$ [DPI]

$\leq \sum_{i=1}^n I(X_i; Y_i, Z_i) + n\mathcal{J}_n$ [memoryless]

$= n I(X_Q; Y_Q, Z_Q \mid Q) + n\mathcal{J}_n$ $[Q \sim \text{Unif}\{1,\dots,n\} \perp\!\!\!\perp (X^n, Y^n, Z^n)]$ (time sharing)

$\leq n I(X_Q; Y_Q, Z_Q) + n\mathcal{J}_n$ $[Q \to X_Q \to (Y_Q, Z_Q)]$

$\Rightarrow R \leq I(X_Q; Y_Q, Z_Q) + \mathcal{J}_n$

$I(X_Q; Y_Q, Z_Q \mid Q) = H(Y_Q, Z_Q \mid Q) - H(Y_Q, Z_Q \mid Q, X_Q) \leq I(X_Q; Y_Q, Z_Q)$

MAC bound:

$nR \leq I(X^n; Y^n) + I(W; X^n \mid Y^n) + n\mathcal{J}_n$

$\leq \sum_{i=1}^n I(X_i; Y_i) + \underbrace{H(W|Y^n)}_{\leq nR_0} - \underbrace{H(W|X^n, Y^n)}_{\geq 0} + n\mathcal{J}_n$ [memoryless]

$\leq \sum_{i=1}^n I(X_i; Y_i) + nR_0 + n\mathcal{J}_n$

$= n I(X_Q; Y_Q \mid Q) + nR_0 + n\mathcal{J}_n$

$\leq n I(X_Q; Y_Q) + nR_0 + n\mathcal{J}_n$

$\Rightarrow R \leq I(X_Q; Y_Q) + R_0 + \mathcal{J}_n$

Take $n \to \infty$, $P_{X_Q}^{(n)} \to P_X$ WLOG by compactness. $P_{XY} \to I(X;Y)$ cont. in disc. case. ▨

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③ Improving Cutset Bound: (Wu-Özgür-Xie '16)

From MAC bound:

$nR \leq n I(X_Q;Y_Q) + H(W|Y^n) - H(W|X^n) + n\mathcal{J}_n$ [Symmetry: $W \perp\!\!\!\perp Y^n | X^n$]

Let $H(W|X^n) = n\varepsilon_n$. First, observe that:

Intuition
1. If $H(W|X^n) \approx 0$, then all Z^n jointly typical with X^n map to W given X^n. Since $P_{Y|X} = P_{Z|X}$, we also expect $H(W|Y^n) \approx 0$. So, $H(W|Y^n) \leq nR_0$ is a loose bound.
2. If $\varepsilon_n \geq \varepsilon$ for large n, then $H(W|X^n) \geq 0$ is a loose bound.

Information inequality: $Y^n \overset{DMC}{\longleftrightarrow} X^n \overset{DMC}{\longleftrightarrow} Z^n \xrightarrow{f_n} W$, $\varepsilon_n = \frac{H(W|X^n)}{n}$. ← Do not need reliable code information.

same

$\forall n \in \mathbb{N}$, $H(W|Y^n) \leq n\, g(\varepsilon_n)$ for some function $g: \mathbb{R}^+ \to \mathbb{R}^+$ continuous with $g(0) = 0$.

explicit examples in [Wu-Özgür-Xie '16]

Then, we have: $nR \leq nI(X_Q;Y_Q) + H(W|Y^n) - n\varepsilon_n + n\mathcal{J}_n$

$H(W|Y^n) \leq nR_0$ [card. bound] $\qquad H(W|Y^n) \leq n\,g(\varepsilon_n)$ [info. ineq.]

$R \leq I(X_Q;Y_Q) + R_0 - \varepsilon_n + \mathcal{J}_n$, $\qquad R \leq I(X_Q;Y_Q) + g(\varepsilon_n) - \varepsilon_n + \mathcal{J}_n$.

So, we get:
$$R \leq I(X_Q;Y_Q,Z_Q) + \mathcal{J}_n$$
$$R \leq I(X_Q;Y_Q) + R_0 - \varepsilon_n + \mathcal{J}_n \quad , \text{ where } 0 \leq \varepsilon_n \leq R_0.$$
$$R \leq I(X_Q;Y_Q) + g(\varepsilon_n) - \varepsilon_n + \mathcal{J}_n$$

By compactness, $\varepsilon_n \to \varepsilon \in [0, R_0]$ because $P_{X_Q}^{(n)} \to P_X$ WLOG as $n \to \infty$. So, letting $n \to \infty$, we derive the following result:

↳ continuity of cond. entropy

Thm: For the symmetric primitive relay channel, if R is achievable, then $\exists P_X, \exists \varepsilon \in [0, R_0]$ s.t.:
$$R \leq I(X;Y,Z),$$
$$R \leq I(X;Y) + R_0 - \varepsilon,$$
$$R \leq I(X;Y) + g(\varepsilon) - \varepsilon.$$

} better than MAC bound: $R \leq I(X;Y) + R_0$

Remarks:
1. Improvement is not obtained via tensorization.
2. Strictly tighter than cutset bound for $R_0 > 0$. [$\varepsilon > 0 \Rightarrow$ obvious, $\varepsilon = 0 \Rightarrow$ Ineq. (3) is $R \leq I(X;Y)$]
3. As in cutset bound, all 3 inequalities are coupled by P_X.

④ Proof of Information Inequality: (special case)

Suppose $Y^n \overset{DMC}{\longleftrightarrow} X^n \overset{DMC}{\longleftrightarrow} Z^n \longrightarrow W = f_n(Z^n) \in \{1, \dots, 2^{nR_0}\}$, and $\frac{1}{n} H(W|X^n) \triangleq \varepsilon_n \xrightarrow{n\to\infty} 0$.

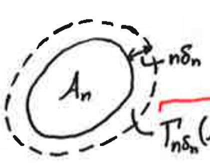
Thm: $H(W|Y^n) \leq n g(\varepsilon_n)$ for some function $g: \mathbb{R}^+ \to \mathbb{R}^+$ s.t. $\underbrace{0 = g(0) = \lim_{x \to 0^+} g(x)}_{\text{continuous at zero}}$, and for sufficiently large n.

Blowing-Up Lemma: (Ahlswede-Gács-Körner '76, Marton '86)

↳ original $\qquad$ ↳ info. theoretic proof (transportation-cost inequality)

Let $U_1, U_2, \dots, U_n \in \mathcal{U}$ be independent random variables. Suppose $A_n \subseteq \mathcal{U}^n$ satisfies $\mathbb{P}(U^n \in A_n) \geq 2^{-n\varepsilon_n}$ for $\varepsilon_n \xrightarrow{n\to\infty} 0$. Then, $\exists$ sequences $\delta_n \xrightarrow{n\to\infty} 0$ and $\eta_n \xrightarrow{n\to\infty} 0$ such that $\mathbb{P}(U^n \in \Gamma_{n\delta_n}(A_n)) \geq 1 - \eta_n$.

only depend on ε_n, not on A_n $\Rightarrow$ can use any A_n with same sequences



$n\delta_n$-blowup of A_n (or) $n\delta_n$-enlargement of A_n

$\Gamma_{n\delta_n}(A_n) \triangleq \{u^n \in \mathcal{U}^n : \exists v^n \in A_n, d_H(u^n, v^n) \leq n\delta_n\}$.

Hamming distance: $d_H(u^n, v^n) = \sum_{i=1}^{n} \mathbb{1}\{u_i \neq v_i\}$

Marton's proof: $\eta_n = \frac{\sqrt{\varepsilon_n}}{\delta_n}$ and choose δ_n s.t. $\eta_n \xrightarrow{n\to\infty} 0$.
Eg: $\delta_n = \varepsilon_n^{1/4}$, $\eta_n = \varepsilon_n^{1/4}$

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Proof: (Polyanskiy '16)

① Lower Bound $P_{W|X^n}$ using Markov's Inequality:

$H(W|X^n) = n\varepsilon_n$

$$\mathbb{P}\left(P_{W|X^n}(W|X^n) < 2^{-n\sqrt{\varepsilon_n}}\right) = \mathbb{P}\left(\log_2\left(\frac{1}{P_{W|X^n}(W|X^n)}\right) > n\sqrt{\varepsilon_n}\right) \leq \frac{H(W|X^n)}{n\sqrt{\varepsilon_n}} = \sqrt{\varepsilon_n} \quad \text{[Markov inequality]}$$

Let $S = \{(w,x^n) \in \{1,\dots,2^{nR_0}\} \times \mathcal{X}^n : P_{W|X^n}(w|x^n) \geq 2^{-n\sqrt{\varepsilon_n}}\}$, and note that:

$\mathbb{P}((W,X^n) \in S) \geq 1 - \sqrt{\varepsilon_n}$. [large n]

② Let $f_n^{-1}(\omega) \triangleq \{z^n \in \mathcal{Y}^n : f_n(z^n) = \omega\}$ for any $\omega \in \{1,\dots,2^{nR_0}\}$. ← pre-image set

Show $\mathbb{P}(Y^n \in T_{n\delta_n}(f_n^{-1}(W)))$ is large using Blowing-Up Lemma:

For $(w,x^n) \in S$, $P_{W|X^n}(w|x^n) = \mathbb{P}(Z^n \in f_n^{-1}(\omega) | X^n = x^n) \geq 2^{-n\sqrt{\varepsilon_n}}$.

$$\xRightarrow[\text{Lemma}]{\text{Blow-up}} \quad \mathbb{P}(Z^n \in T_{n\delta_n}(f_n^{-1}(\omega)) | X^n = x^n) \geq 1 - \eta_n, \quad \text{where } \delta_n = \eta_n = \varepsilon_n^{\frac{1}{8}}.$$

$$\xRightarrow[\text{Symmetry}]{P_{Y|X} \overset{d}{=} P_{Z|X}} \quad \mathbb{P}(Y^n \in T_{n\delta_n}(f_n^{-1}(\omega)) | X^n = x^n) \geq 1 - \eta_n$$

$$\mathbb{P}(Y^n \in T_{n\delta_n}(f_n^{-1}(W))) \geq \sum_{(\omega,x^n)\in S} \underbrace{\mathbb{P}(Y^n \in T_{n\delta_n}(f_n^{-1}(\omega)) | X^n = x^n, \cancel{W=\omega})}_{\geq 1-\eta_n} P_{W,X^n}(\omega,x^n) \qquad Y^n \perp\!\!\!\perp W | X^n$$

$$\geq (1-\eta_n)\,\mathbb{P}((W,X^n)\in S)$$

$$\geq (1-\eta_n)(1-\sqrt{\varepsilon_n}) \quad \leftarrow \eta_n = \varepsilon_n^{\frac{1}{8}}$$

$$\geq 1 - 2\varepsilon_n^{\frac{1}{8}}. \quad \text{[large n]}$$

③ Bound $H(W|Y^n)$ using List Decoding + Fano's Inequality:
↳(Ahlswede – Dueck '76)

Let $E = \mathbb{1}\{Y^n \in T_{n\delta_n}(f_n^{-1}(W))\}$.
↑indicator r.v.

$$H(W|Y^n) \leq H(W,E|Y^n) = H(E|Y^n) + H(W|E,Y^n)$$

$$\leq \underbrace{H(E)}_{\leq h(2\varepsilon_n^{1/8})} + \underbrace{P_E(0)}_{\leq 2\varepsilon_n^{1/8}}\underbrace{H(W|E=0,Y^n)}_{\leq nR_0} + \underbrace{P_E(1)}_{\approx 1}\underbrace{H(W|E=1,Y^n)}$$

[large n] ← (binary entropy)

If $Y^n \in T_{n\delta_n}(f_n^{-1}(W))$, then $\exists y^n \in f_n^{-1}(\omega) \cap \underbrace{\{z^n \in \mathcal{Y}^n : d_H(z^n,y^n) \leq n\delta_n\}}_{\text{Hamming Ball}_{y^n}(n\delta_n)}$.

$$\Rightarrow H(W|Y^n, E=1) \leq \log_2\left(\left|\{z^n \in \mathcal{Y}^n : d_H(z^n,y^n) \leq n\delta_n\}\right|\right)$$

$$\leq n\left(\delta_n \log_2(|\mathcal{Y}|-1) + h(\delta_n)\right)$$

↑bound on volume of Hamming ball (requires $n\delta_n \geq 1$)

$\delta_n = \varepsilon_n^{\frac{1}{8}}$

$$\Rightarrow H(W|Y^n) \leq h(2\varepsilon_n^{\frac{1}{8}}) + n\,2\varepsilon_n^{\frac{1}{8}}R_0 + n\left(\delta_n\log_2(|\mathcal{Y}|-1) + h(\delta_n)\right)$$

$$\Rightarrow H(W|Y^n) \leq n\underbrace{\left(h(2\varepsilon_n^{\frac{1}{8}}) + 2\varepsilon_n^{\frac{1}{8}}R_0 + h(\varepsilon_n^{\frac{1}{8}}) + \varepsilon_n^{\frac{1}{8}}\log_2(|\mathcal{Y}|-1)\right)}_{\triangleq g(\varepsilon_n)\ (\to 0 \text{ as } \varepsilon_n \to 0)}$$

$\therefore$ $H(W|Y^n) \leq n g(\varepsilon_n)$.

■

Remarks:

1. Bound better for larger n.
2. Such bound does not hold without factoring $X^n \to Z^n \to W$. [Eg: $Y_n \leftarrow X^n \to W$. Let $W = X^n$, $Y^n \perp\!\!\!\perp X^n$. Then, $H(W|X^n) = 0$, $H(W|Y^n) = H(X^n)$.]

THE END

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Volume of Hamming Ball:

Consider the set $\mathcal{Y}$ with $2 \leq |\mathcal{Y}| < +\infty$. For $y^n, z^n \in \mathcal{Y}^n$, let $d_H(y^n, z^n) \triangleq \sum_{i=1}^{n} \mathbb{1}\{y_i \neq z_i\}$ denote Hamming distance.

Let $\text{Ball}_{y^n}(nr) \triangleq \{z^n \in \mathcal{Y}^n : d_H(y^n, z^n) \leq nr\}$ denote the Hamming ball of radius nr around $y^n \in \mathcal{Y}^n$.

Prop: For $0 \leq r \leq 1 - \frac{1}{|\mathcal{Y}|}$, and sufficiently large n (s.t. $nr \geq 1$), we have:

$$\frac{1}{n}\log\left(|\text{Ball}_{y^n}(nr)|\right) \leq r\log(|\mathcal{Y}|-1) + h(r).$$

[Note: This is asymptotically tight.]

any base

Proof:

$$\begin{aligned}
1 &= (r + (1-r))^n \\
&= \sum_{i=0}^{n} \binom{n}{i} r^i (1-r)^{n-i} \\
&\geq \sum_{i=0}^{\lfloor nr \rfloor} \binom{n}{i} r^i (1-r)^{n-i} \\
&= \sum_{i=0}^{\lfloor nr \rfloor} \binom{n}{i} (|\mathcal{Y}|-1)^i (1-r)^n \left(\frac{r}{(|\mathcal{Y}|-1)(1-r)}\right)^i \\
&\geq (1-r)^n \left(\frac{r}{(|\mathcal{Y}|-1)(1-r)}\right)^{nr} \underbrace{\sum_{i=0}^{\lfloor nr \rfloor} \binom{n}{i} (|\mathcal{Y}|-1)^i}_{= |\text{Ball}_{y^n}(nr)|}
\end{aligned}$$

$\left[\frac{r}{(|\mathcal{Y}|-1)(1-r)} \leq 1 \Leftrightarrow r \leq |\mathcal{Y}| - 1 - r(|\mathcal{Y}|-1) \Leftrightarrow \underbrace{r|\mathcal{Y}| \leq |\mathcal{Y}|-1}_{\text{this is assumed}}\right]$

$$= |\text{Ball}_{y^n}(nr)| (1-r)^n \left(\frac{r}{(|\mathcal{Y}|-1)(1-r)}\right)^{nr} = |\text{Ball}_{y^n}(nr)|\, r^{nr} (1-r)^{n(1-r)} (|\mathcal{Y}|-1)^{-nr}$$

$$\Rightarrow 0 \geq \frac{1}{n}\log\left(|\text{Ball}_{y^n}(nr)|\right) + r\log(r) + (1-r)\log(1-r) - r\log(|\mathcal{Y}|-1)$$

$$\Rightarrow \frac{1}{n}\log\left(|\text{Ball}_{y^n}(nr)|\right) \leq r\log(|\mathcal{Y}|-1) + h(r)$$

binary entropy

$\blacksquare$

Intuition:

$$|\text{Ball}_{y^n}(nr)| \leq \underbrace{(|\mathcal{Y}|-1)^{nr}}_{\substack{\text{no. of choices} \\ \text{of replacement} \\ \text{at } nr \text{ locations}}} \cdot \underbrace{\exp(nh(r))}_{\substack{\text{no. of binary strings of length } n \\ \text{with type } (nr\ 1\text{'s},\ n(1-r)\ 0\text{'s})}}$$

[nr distance dominates lower distance strings]

encode changed letters